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COUPLED SYSTEMS SUBJECTED TO DETERMINATE AND RANDOM INPUT

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Abstract—A method is presented for the analysis of complex structural systems which can be subdivided into any number of component systems arbitrarily interconnected at discrete points. Using experimentally or analytically determined receptances (frequency response functions) characterizing the mechanical properties of the component systems, receptances are derived which characterize the mechanical properties of the entire coupled system. The requirement of system continuity at the coupling points gives rise to conditions of equilibrium and compatibility at the connections. These conditions are modified allowing for the presence of elastic and/or dissipative coupling units with negligible masses between the coupling points, thus, adding considerable practical flexibility to the method. Keeping the contributions of the individual component systems identified, it is then shown how the receptances enter response calculations for the entire system which is subjected to determinate or random excitations.

NOTATION

$A, B, C, \ldots N$	as superscripts indicate component system
$a, b, c, \ldots n$	as subscripts indicate points
$\overline{M}_m$	mth constrained force
K_m	mth coupling unit
D_j^J	constrained displacement in system J at point j
M_j^J	constrained force in system J at point j
P_j^J	steady state excitation in system J at point j
$p_j^J(t)$	excitation time function
X_j^J	steady state response in system J at point j
$x_j^J(t)$	response time function
$\bar{\bar{\theta}}_{mk}$	receptance between coupling points in system J
$\bar{\theta}_{mk}$	receptance between a coupling point and an excitation point in system J
$*\bar{\theta}_{mk}$	receptance between a coupling point and a response point in system J
θ_{mk}	receptance between two field points in system J
$\mathscr{D}$	receptance matrix
[ ]	rectangular matrix
$[^T]$	transpose of rectangular matrix
{ }	column matrix
$\overset{JK}{\underset{jk}{H}}(\omega)$	receptance giving the response at point j in component system J due to a steady state excitation with frequency ω at point k in component system K
$h_{jk}^{JK}(t-\tau)$	unit impulse function giving the response at point j in component system J due to a unit impulse excitation at time $(t-\tau)$ at point k in component system K
$\overset{JK}{\underset{jk}{R}}*(\tau)$	response cross-correlation function
$\overset{JK}{\underset{jk}{R}}p(\tau)$	excitation cross-correlation function

$\overset{JK}{\underset{jk}{S}}{}^{*}(\omega)$	response cross-power spectral density
$\overset{JK}{\underset{jk}{S}}\,p(\omega)$	excitation cross-power spectral density
ω	angular frequency
t	time
τ	time delay

INTRODUCTION

WHILE methods of analysis, applicable to simple structural elements, have been known and successfully applied for well over a century, during recent years, analysts have directed their attention to the analysis and evaluation of the general dynamical and vibrational behavior of complex structural systems such as high rise buildings, missiles, aeroplanes, space stations, etc. These systems are frequently too involved to be amenable to proper idealizations and thus to purely analytical treatment. It becomes therefore necessary to resort to experimental vibration techniques which may be carried out on scaled model structures or on the actual full-size structure. Because of difficulties in proper scaling and simulation, vibration tests on models frequently offer less chance of successful predictions than analytical idealizations. The most reliable results are to be expected, therefore, from properly conducted full-size tests. This is not only expensive, but frequently infeasible since the size of the structural system may be such that it cannot be tested in one piece, or different parts of the whole system are fabricated and assembled at different geographical locations.

This then requires a method by which knowledge of certain measured vibrational characteristics of the component parts of the system allows the prediction of the vibrational characteristics of the whole system.

In more recent years, several methods have been devised by which the behavior of a structural system may be predicted in terms of the properties of its components. The method presented in this paper falls into this category, and in order to clarify the position of this study relative to other studies in this area, reference is made to some typical ones that have appeared in the recent literature.

A method of analysis of considerable accuracy and versatility has grown out of the well-known methods of Holzer [1] and Myklestad [2]. It has been generalized by Pestel and associates [3–5] and is known as the method of "transfer matrices". A method based on component mode analysis has been presented by Hurty [6]. Another powerful method of analysis employs the concept of "receptance", also called frequency response function, mechanical admittance, or dynamic influence coefficient. Duncan [7] has given an account of its application to simple conservative systems under the action of isolated harmonic forces and has extended it to simple systems with damping and aerodynamic actions. Based on this work, Bishop and Johnson [8] extensively applied the concept of receptance to various simple structures.

In most of these works, damping is neglected altogether, or it is dealt with in an idealized manner with the assumption that it is small enough so that every mode of vibration may be considered a single oscillator. A notable exception to this, presented by Hurty [6], allows for damping at discrete points that is neither small nor of such a nature as to prevent coupling of the undamped natural modes. An extensive discussion of systems with a finite number of degrees of freedom subject to linear damping, together with an assessment of various resonance testing techniques, has been given by Bishop and Gladwell [9]. All of these investigations are based on the determination of eigen frequencies and modal shapes which usually poses the major problem.

In this paper, the concept of mode is, therefore, discarded in favor of the concept of receptance. In this context, a method is devised by which the receptance measurements on the component systems may be translated automatically by digital means into receptances of the entire system. It is then shown how this is adaptable to response calculations for the entire system which is subjected to determinate or random inputs. Allowance is made for any number of component systems which are interconnected to each other at any arbitrary number of points. In addition, elastic and/or dissipative coupling units at the coupling points are considered, thus simplifying the experimental measurements and allowing to vary the coupling conditions without repeating the determination of receptances of the component systems. No assumptions are made with regard to the magnitude of damping in the component system and in the coupling units.

SYSTEM REPRESENTATION

When more than two component systems are coupled, it is usually expedient to set up the governing equations with the help of a block diagram, Fig. 1.

(i) Component systems are represented by rectangles $A, B, C, \ldots N$.

(ii) Coupling points in each system are numbered consecutively $1, 2, \ldots a$, in A; $1, 2, \ldots b$, in B; $\ldots, 1, 2, \ldots n$, in N. Two or more independent couplings at the same physical point are counted as two or more coupling points, a situation occurring, e.g. at points where forces as well as moments are transmitted.

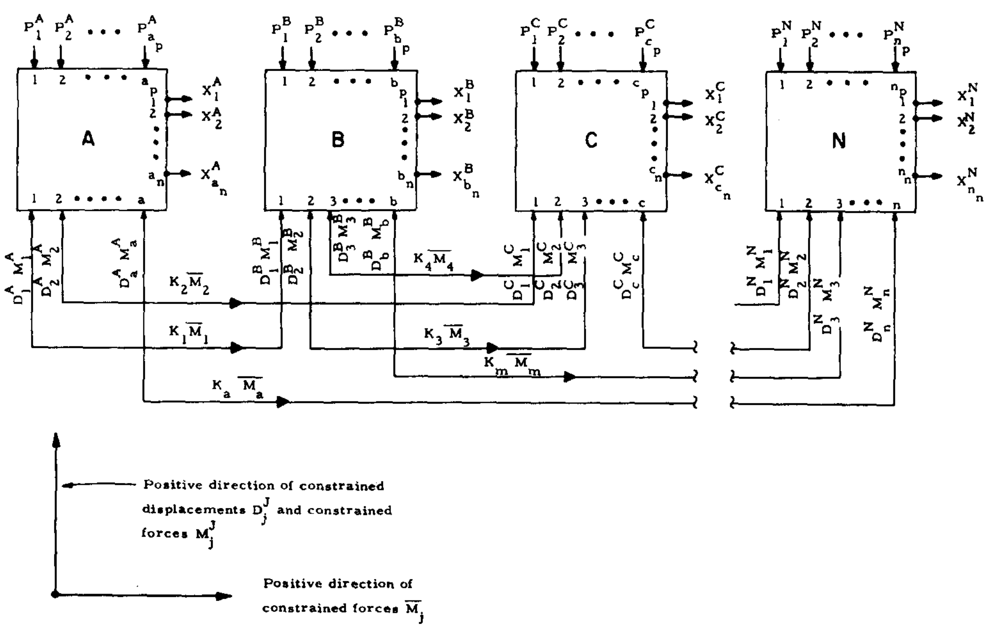

FIG. 1. Representation of coupled system.

(iii) Constrained forces $M_1^A, \dots M_a^A$, in A; $M_1^B, \dots M_b^B$, in B; $\dots, M_1^N, \dots M_n^N$, in N; and constrained displacements $D_1^A, \dots D_a^A$, in A; $D_1^B, \dots D_b^B$, in B; $\dots, D_1^N, \dots D_n^N$, in N; are represented as positive when pointing into the component systems.

(iv) The appropriate coupling points are connected in accordance with the actual physical system. The connecting lines indicate the positive direction of the constrained forces $\overline{M}_1, \overline{M}_2, \dots \overline{M}_m, \dots$ and represent the corresponding coupling units $K_1, K_2, \dots K_m, \dots$.

(v) External excitations, positive when pointing into the system, are designated consecutively by $P_1^A, \dots P_{ap}^A$, in A; $P_1^B, \dots P_{bp}^B$, in B; $\dots, P_1^N, \dots P_{np}^N$, in N, where the subscripts indicate the point of application.

In Fig. 1, all impressed excitations are, with respect to a common origin, the real parts of the steady state amplitudes of these quantities which vary sinusoidally with a particular frequency ω. Thus, if the amplitude of the excitation in a typical system J, at a typical point j, is given by $\bar{P}_j^J(\omega)$, the corresponding component in Fig. 1 may be written in the usual complex form

$$P_j^J(\omega) = \bar{P}_j^J(\omega)\exp(\Phi_j^J(\omega)). \tag{1}$$

Similar remarks apply to the response $X_j^J(\omega)$. Hence, all quantities introduced in Fig. 1 are in general steady state complex numbers at frequency ω.

THE SYSTEM MATRIX

In the typical component system J with the given external steady state excitations $P_1^J, P_2^J, \dots P_{jp}^J$ and the as yet unknown steady state constrained forces at the coupling points, $M_1^J, M_2^J, \dots M_j^J$, the steady state response at every point of the component system J is obtained using the predetermined receptances at frequency ω. At a typical coupling point m, the steady state constrained displacement is

$$D_m^J = \sum_{k=1}^{j} \bar{\bar{\theta}}_{mk} M_k^J + \sum_{k=1}^{jp} \bar{\theta}_{mk} P_k^J \tag{2}$$

where $\bar{\bar{\theta}}_{mk}$ are receptances between coupling points, and $\bar{\theta}_{mk}$ are receptances involving only one coupling point. The typical receptance θ_{jk} is defined as the ratio between the steady state displacement-like response at any point j and the force-like excitation with constant frequency ω at any point k. Because of the ever present damping, there exists a phase lag between the amplitudes of excitation and response. The receptances, therefore, appear in general as complex numbers, where both the real and imaginary part, or equivalently the modulus and phase angle are given as functions of frequency ω.

The constrained displacement vector at frequency ω of the coupling points in system J follows from equation (2) as

$$\begin{Bmatrix} D_1^J \\ D_2^J \\ \vdots \\ D_m^J \\ \vdots \\ D_j^J \end{Bmatrix} = \begin{bmatrix} \bar{\bar{\theta}}_{11} & \bar{\bar{\theta}}_{12} & \dots & \bar{\bar{\theta}}_{1j} \\ \bar{\bar{\theta}}_{21} & \bar{\bar{\theta}}_{22} & & \vdots \\ \vdots & \ddots & & \vdots \\ \vdots & & \bar{\bar{\theta}}_{mm} & \vdots \\ \vdots & & \ddots & \vdots \\ \bar{\bar{\theta}}_{j1} & \dots & \dots & \bar{\bar{\theta}}_{jj} \end{bmatrix} \begin{Bmatrix} M_1^J \\ M_2^J \\ \vdots \\ M_m^J \\ \vdots \\ M_j^J \end{Bmatrix} + \begin{bmatrix} \bar{\theta}_{11} & \bar{\theta}_{12} & \dots & \bar{\theta}_{1jp} \\ \bar{\theta}_{21} & \bar{\theta}_{22} & & \vdots \\ \vdots & \ddots & & \vdots \\ \vdots & & \bar{\theta}_{mm} & \vdots \\ \vdots & & \ddots & \vdots \\ \bar{\theta}_{j1} & & \bar{\theta}_{jj} \dots & \theta_{jjp} \end{bmatrix} \begin{Bmatrix} P_1^J \\ P_2 \\ \vdots \\ \vdots \\ \vdots \\ P_{jp}^J \end{Bmatrix} \tag{3}$$

or, changing to symbolic matrix notation,

$$\{D^J\} = [\bar{\mathscr{D}}^J]\{M^J\}+[\mathscr{D}^J]\{P^J\}. \tag{4}$$

Similar steady state equations can be written for every component system and may be combined in the following array.

$$\begin{Bmatrix} \{D^A\} \\ \{D^B\} \\ \vdots \\ \{D^J\} \\ \vdots \\ \{D^N\} \end{Bmatrix} = \begin{bmatrix} \bar{\mathscr{D}}^A & & & & \\ & \bar{\mathscr{D}}^B & & & \\ & & \ddots & & \\ & & & \bar{\mathscr{D}}^J & \\ & & & & \ddots \\ & & & & & \bar{\mathscr{D}}^N \end{bmatrix} \begin{Bmatrix} \{M^A\} \\ \{M^B\} \\ \vdots \\ \{M^J\} \\ \vdots \\ \{M^N\} \end{Bmatrix} + \begin{bmatrix} \mathscr{D}^A & & & & \\ & \mathscr{D}^B & & & \\ & & \ddots & & \\ & & & \mathscr{D}^J & \\ & & & & \ddots \\ & & & & & \mathscr{D}^N \end{bmatrix} \begin{Bmatrix} \{P^A\} \\ \{P^B\} \\ \vdots \\ \{P^J\} \\ \vdots \\ \{P^N\} \end{Bmatrix} \tag{5}$$

or, with obvious change of notation,

$$\{D\} = [\bar{\mathscr{D}}]\{M\}+[\mathscr{D}]\{P\}. \tag{6}$$

To reduce the uncoupled equations (5) or (6) to a coupled solvable system, supplementary coupling conditions, expressing compatibility and equilibrium at the coupling points, must be prescribed. Because of practical design considerations, which in most cases require deformable rather than rigid connections at the coupling points, it is expedient to introduce coupling units between the test points of two systems at which their receptances are measured, Fig. 2. These coupling units are considered in the coupling conditions in which they simulate mathematically, as complex functions of frequency ω, the deformational behavior of such items as springs, bolts, washers, cushionings, etc.

In the coupled system, the test points perform motions relative to each other in accordance with the constrained forces M_m which are transmitted through the coupling unit. Because of assumed linearity, the relative displacement of the test points is proportional to M_m, and since the coupling unit is in general elastic as well as dissipative, the proportionality constant $K_m(\omega)$ is in general complex and frequency dependent. Hence, the relative displacements between the test points are

$$d_m = K_m(\omega)M_m. \tag{7}$$

If the test points are connected directly to each other, then $d_m = 0$, i.e. $K_m = 0$.

With the help of Fig. 1 and equation (7), the conditions of compatibility can be written

$$[C]\{D\} = [\diagdown K \diagdown]\{\bar{M}\}. \tag{8}$$

If the column vector $\{D\}$ in equation (8) includes $2k$ elements, then the coupling matrix $[C]$ is a $k\times 2k$ matrix which has only two non-zero elements in each row, (-1) and $(+1)$.

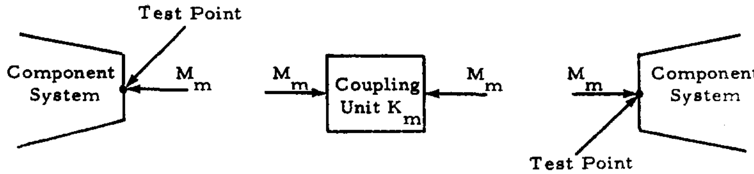

FIG. 2. Typical coupling unit.

$\lceil K \rfloor$ is a $k \times k$ diagonal constrained matrix which includes the coupling units K_m, while the column matrix $\{\overline{M}\}$ includes the corresponding constrained forces.

Again, following Fig. 1, it is easily verified that the equilibrium conditions can be written as

$$[\overset{T}{C}]\{\overline{M}\} = \{M\} \tag{9}$$

where $[\overset{T}{C}]$ is the transpose of the coupling matrix in equation (8).

Premultiplying equation (6) by $[C]$ and substituting equations (8) and (9), one obtains for the unknown constrained forces,

$$\left[\lceil K \rfloor \quad -[C][\overline{\mathscr{D}}][\overset{T}{C}]\right]\{\overline{M}\} = [C][\mathscr{D}]\{P\}. \tag{10}$$

If the external excitations vanish, equation (10) gives the equations of free vibrations of the entire coupled system, and the constrained forces may be different from zero only, if the coefficient determinant vanishes, i.e.

$$\left|\lceil K \rfloor \quad -[C][\overline{\mathscr{D}}][\overset{T}{C}]\right| = 0. \tag{11}$$

If damping is present, the roots of equation (11) are, in general, complex. For very light damping, the imaginary parts of these roots are approximately equal to the eigenfrequencies of free vibrations of the same undamped system, and the displacements of the entire system at these frequencies are approximately equal to the free vibration modes.

Premultiplying equation (10) with the inverse of its coefficient determinant, and then with $[\overset{T}{C}]$ one obtains

$$\{M\} = [T][\mathscr{D}]\{P\} \tag{12}$$

where

$$[T] = \left[[\overset{T}{C}]\left[\lceil K \rfloor \quad -[C][\overline{\mathscr{D}}][\overset{T}{C}]\right]^{-1}[C]\right] \tag{13}$$

is a $2k \times 2k$ matrix which characterizes the mechanical properties of the entire coupled system in terms of the mechanical properties of the component systems.

Expanded, the system matrix $[T]$ becomes

$$[T] = \begin{bmatrix} t_{11}^{AA} & & t_{11}^{AB} & & \dots t_{11}^{AN} & \\ & \ddots & & \ddots & & \ddots \\ & t_{aa}^{AA} & & t_{ab}^{AB} & & t_{an}^{AN} \\ t_{11}^{BA} & & t_{11}^{BB} & & & \\ & \ddots & & \ddots & & \\ & t_{ba}^{BA} & & t_{bb}^{BB} & & \\ \vdots & & \vdots & & \ddots & \\ t_{11}^{NA} & & & & t_{11}^{NN} & \\ & \ddots & & & & \ddots \\ & t_{na}^{NA} & & & & t_{mn}^{NN} \end{bmatrix} = \left[\begin{array}{c|c|c|c} T^{AA} & T^{AB} & \dots & T^{AN} \\ \hline T^{BA} & T^{BB} & \dots & T^{BN} \\ \hline \vdots & \vdots & \ddots & \\ \hline T^{NA} & T^{NB} & \dots & T^{NN} \end{array}\right] \tag{14}$$

where the partitioning is the same as in $[\bar{\mathscr{D}}]$, equation (5). Once the elements of $[T]$ have been determined, they may be recorded for future use. All or part of $[T]$ is used in subsequent computations depending on the information one intends to obtain.

RESPONSE TO DETERMINATE EXCITATIONS

The steady state response in the typical component system J at the typical point j is

$$X_j^J(\omega) = \sum_k \theta_{jk}(\omega) P_k^J(\omega) + \sum_k {}^*\bar{\theta}_{jk}(\omega) M_k^J(\omega) \tag{15}$$

where the $\theta_{jk}(\omega)$ are receptances between response points j and excitation points k not involving coupling points. The ${}^*\bar{\theta}_{jk}(\omega)$ are receptances involving one response point indicated by a star and one coupling point indicated by a bar. In matrix terminology, the response vector of component system J is then

$$\{X^J\} = [\mathscr{D}^J]\{P^J\} + [{}^*\bar{\mathscr{D}}^J]\{M^J\}. \tag{16}$$

The constrained force vector $\{M^J\}$ can be eliminated from equation (16) with the help of the corresponding submatrix in $\{M\}$ of equation (12). Expanding equation (12) and using equation (14), one obtains

$$\{M^J\} = [T^{JA}][\mathscr{D}^A]\{P^A\} \ldots + [T^{JJ}][\mathscr{D}^J]\{P^J\} \ldots + [T^{JN}][\mathscr{D}^N]\{P^N\} \tag{17}$$

with similar equations for $\{M^A\} \ldots \{M^N\}$. Substituting equation (17) in equation (16), the steady state response at frequency ω of the entire coupled system becomes

$$\begin{Bmatrix} \{X^A\} \\ \vdots \\ \{X^J\} \\ \vdots \\ \{X^N\} \end{Bmatrix} = \begin{bmatrix} \left[[\mathscr{D}^A] + [{}^*\bar{\mathscr{D}}^A][T^{AA}][\mathscr{D}^A]\right] \ldots [{}^*\bar{\mathscr{D}}^A][T^{AJ}][\mathscr{D}^J] \ldots [{}^*\bar{\mathscr{D}}^A][T^{AN}][\mathscr{D}^N] \\ [{}^*\bar{\mathscr{D}}^J][T^{JA}][\mathscr{D}^A] \ldots \left[[\mathscr{D}^J] + [{}^*\bar{\mathscr{D}}^J][T^{JJ}][\mathscr{D}^J]\right] \ldots [{}^*\bar{\mathscr{D}}^J][T^{JN}][\mathscr{D}^N] \\ [{}^*\bar{\mathscr{D}}^N][T^{NA}][\mathscr{D}^A] \ldots [{}^*\bar{\mathscr{D}}^N][T^{NJ}][\mathscr{D}^J] \ldots \left[[\mathscr{D}^N] + [{}^*\bar{\mathscr{D}}^N][T^{NN}][\mathscr{D}^N]\right] \end{bmatrix}$$

$$\times \begin{Bmatrix} \{P^A\} \\ \vdots \\ \{P^J\} \\ \vdots \\ \{P^N\} \end{Bmatrix} \tag{18}$$

or

$$\{X(\omega)\} = [H(\omega)]\{P(\omega)\} \tag{19}$$

where the typical elements of the typical submatrix $[H^{JK}]$ of the receptance matrix $[H]$ in equation (19) are the receptances of the coupled system in terms of the receptances of the component systems and are designated by $H_{jk}^{JK}(\omega)$, giving the response at point j in component system J due to a steady state excitation with frequency ω at point k in component system K.

The response of the coupled system in the time domain is obtained from the response in the frequency domain, equation (19), by the use of the Fourier transform pair,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{i\omega\tau}\,d\omega \tag{20a}$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}\,dt. \tag{20b}$$

Applying equation (20a) to equation (19) and equation (20b) to $\{P(\omega)\}$ with $P(\omega)$ replacing $X(\omega)$ and $p(t)$ replacing $x(t)$, one obtains

$$\{x(t)\} = \int_{\tau=-\infty}^{\tau=\infty}\left(\frac{1}{2\pi}\int_{\omega=-\infty}^{\omega=\infty}[H(\omega)]\,e^{i\omega(t-\tau)}\,d\omega\right)\{p(\tau)\}\,d\tau \tag{21}$$

where the interchangeability of the order of integration has been assumed. When the unit impulse matrix is defined in terms of the receptance matrix by the Fourier transform relation,

$$[h(t-\tau)] = \frac{1}{2\pi}\int_{-\infty}^{\infty}[H(\omega)]e^{i\omega(t-\tau)}\,d\omega \tag{22}$$

the response equation in the time domain becomes

$$\{x(t)\} = \int_{-\infty}^{\infty}[h(t-\tau)]\{p(\tau)\}\,d\tau \tag{23}$$

where the integrals in equations (22) and (23) are to be taken over every product term under the integral sign. Equation (23) in the time domain is analogous to equation (19) in the frequency domain. Similar to the submatrices $[H^{JK}]$ with elements H^{JK}_{jk} in the frequency domain, the unit impulse matrix, equation (22), is subdivided in submatrices $[h^{JK}(t-\tau)]$ with unit impulse functions $h^{JK}_{jk}(t-\tau)$ as elements which give the response at point j in component system J due to a unit impulse excitation at time $(t-\tau)$ at point k in component system K.

RESPONSE TO RANDOM EXCITATIONS

It is quite common for modern structural systems to be subjected to random rather than determinate excitations. For instance, in certain geographical areas buildings are excited by earthquake motions which are random in nature. Missile structures and space stations may be under the influence of atmospheric turbulence, boundary layer turbulence, thrust noise, docking excitations, internal operational noise, etc., all of which fall in the category of random excitations.

In the following, relations are presented which govern the response of the coupled system to random excitations when the receptances of the component systems are available as functions of frequency ω.

The cross-correlation functions of the stationary random responses of the coupled system at point j in component system J and at point k in component system K are defined by the following expression [10]

$$\overset{JK}{\underset{jk}{R*}}(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x_j^J(t)x_k^K(t+\tau)\,\mathrm{d}t \tag{24}$$

where τ is a time delay with respect to t and the response $x(t)$ is assumed to be zero outside the "long" time interval T.

The cross-power spectral density $\overset{JK}{\underset{jk}{S*}}(\omega)$ of the response at point j in component system J and at point k in component system K is defined to be the Fourier transform of the cross-correlation function in equation (24). Hence, the following Fourier transform pair holds

$$\overset{JK}{\underset{jk}{R*}}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overset{JK}{\underset{jk}{S*}}(\omega)e^{i\omega\tau}\,\mathrm{d}\omega \tag{25}$$

$$\overset{JK}{\underset{jk}{S*}}(\omega) = \int_{-\infty}^{\infty} \overset{JK}{\underset{jk}{R*}}(\tau)e^{-i\omega\tau}\,\mathrm{d}\tau. \tag{26}$$

Substituting equation (24) in equation (26), one obtains with the help of equation (20b),

$$\overset{JK}{\underset{jk}{S*}}(\omega) = \lim_{T\to\infty} \frac{1}{T}(X_{j_T}^{J*}(\omega)X_{k_T}^{K}(\omega)) \tag{27}$$

where $X_{j_T}^{J*}(\omega)$ is the complex conjugate of $X_{j_T}^{J}(\omega)$, and where for practical reasons the subscript T indicates that in the time domain the response record is zero outside the time interval T.

Equations analogous to the response equations (24), (25), (26) and (27) hold also for the excitations and are given below by equations (28), (29), (30) and (31), respectively.

$$\overset{LM}{\underset{lm}{Rp}}(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} p_l^L(t)p_m^M(t+\tau)\,\mathrm{d}t \tag{28}$$

$$\overset{LM}{\underset{lm}{Rp}}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overset{LM}{\underset{lm}{Sp}}(\omega)e^{i\omega\tau}\,\mathrm{d}\omega \tag{29}$$

$$\overset{LM}{\underset{lm}{Sp}}(\omega) = \int_{-\infty}^{\infty} \overset{LM}{\underset{lm}{Rp}}(\tau)e^{-i\omega\tau}\,\mathrm{d}\tau \tag{30}$$

$$\overset{LM}{\underset{lm}{Sp}}(\omega) = \lim_{T\to\infty} \frac{1}{T}(P_{l_T}^{L*}(\omega)P_{m_T}^{M}(\omega)). \tag{31}$$

The response cross-correlation functions and the response cross-power spectral densities may be determined in terms of the excitation cross-correlation functions and the excitation cross-power spectral densities by substituting equation (23) in equation (24).

Changing from matrix notation to index notation, one then obtains

$$\underset{jk}{\overset{JK}{R*}}(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\left\{\left(\sum_{l=1}^{l_a}\int_{-\infty}^{\infty} h_{jl}^{JA}(\bar{\tau})p_l^A(t-\bar{\tau})\,d\bar{\tau}\ldots\right)\right.$$
$$\left.\times\left(\ldots\sum_{m=1}^{m_n}\int_{-\infty}^{\infty} h_{km}^{KN}(\bar{\bar{\tau}})p_m^N(t+\tau-\bar{\bar{\tau}})\,d\bar{\bar{\tau}}\right)\right\}dt. \tag{32}$$

Multiplying out, exchanging the order of integration and summation and observing equation (28) with a change of variable, the response cross-correlation functions become in terms of the excitation cross-correlation functions

$$\underset{jk}{\overset{JK}{R*}}(\tau) = \sum_{l=1}^{l_p}\sum_{m=1}^{m_p}\sum_{L=A}^{N}\sum_{M=A}^{N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_{jl}^{JL}(\bar{\tau})h_{km}^{KM}(\bar{\bar{\tau}})\underset{lm}{\overset{LM}{Rp}}(\tau+\bar{\tau}-\bar{\bar{\tau}})\,d\bar{\tau}\,d\bar{\bar{\tau}}. \tag{33}$$

Changing in equation (29) the variable τ to $(\tau+\bar{\tau}-\bar{\bar{\tau}})$ and substituting in equation (33) gives

$$\underset{jk}{\overset{JK}{R*}}(\tau) = \sum_{l=1}^{l_p}\sum_{m=1}^{m_p}\sum_{L=A}^{N}\sum_{M=A}^{N}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} h_{jl}^{JL}(\bar{\tau})e^{i\omega\bar{\tau}}\,d\bar{\tau}\int_{-\infty}^{\infty} h_{km}^{KM}(\bar{\bar{\tau}})e^{-i\omega\bar{\bar{\tau}}}\,d\bar{\bar{\tau}}\underset{lm}{\overset{LM}{Sp}}(\omega)\right\}e^{i\omega\tau}\,d\omega. \tag{34}$$

With the inverse of the Fourier transform relations for unit impulse functions, equation (24), and the Fourier transform relations equations (25) and (26), equation (34) becomes

$$\underset{jk}{\overset{JK}{S*}}(\omega) = \sum_{l=1}^{l_p}\sum_{m=1}^{m_p}\sum_{L=A}^{N}\sum_{M=A}^{N} H_{jl}^{JL*}(\omega)H_{km}^{KM}(\omega)\underset{lm}{\overset{LM}{Sp}}(\omega). \tag{35}$$

This equation gives the response cross-power spectral density between point j in component system J and point k in component system K. The right-hand terms are: the complex conjugate of the receptance between point j in component system J and points l in component systems L; the receptance between point k in component system K and points m in component systems M; the excitation cross-power spectral density between points l in component system L and points m in component systems M.

It is clear that the response auto-correlation function and the response power spectral density at point j in component system J is obtained from equation (33) and equation (35), respectively by setting $k = j$ and $K = J$ in these equations.

CONCLUSIONS

A method is presented by which knowledge of the characteristic mechanical properties of component systems allows the determination of the dynamic properties of the coupled system. This method, which does not entail the calculation or measurement of the roots of the characteristic equations and their associated modal shapes, is based on the use of experimentally determined frequency response functions (receptances) of the component systems. The method has thus the important merit in that it may be used when the differential equations for certain components or for the entire system are unknown and the starting data for the response calculations, transient or random, consist of the experimentally determined receptances of the component systems. The method is, of course, not limited to the use of experimentally determined receptances, i.e. analytically determined receptances as functions of frequency may be used as well. The method allows diversified modifications and variations depending on the particular problem at hand.

The receptances of the entire coupled system are given by the elements of the coefficient submatrices in equation (20) and are combinations of the receptances of the component systems. These combinations are most easily executed by standard operations on digital computers for frequency ranges of interest.

With the external excitations given in terms of their spectral densities, the spectral densities of the responses are determined by simple matrix multiplication through the frequency ranges of interest, equation (20).

The use of receptances lends itself naturally to the development of the random response characteristics as expressed by cross-power spectral densities, equation (35), which are necessary for the analysis of structures subjected to random loads.

The method presented here is strictly valid only for coupling of component systems at discrete points. However, by a straightforward extension using Fourier series expansions, coupling of component systems at continuously connected interfaces may be affected along similar lines.

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Résumé—Une méthode est présentée pour l'analyse de systèmes structuraux complexes pouvant être subdivisés en n'importe quel nombre de systèmes composants arbitrairement interreliés à des points discrets. En employant des réceptions déterminées expérimentalement ou analytiquement (fonctions de caractéristique de fréquence) caractérisant les propriétés mécaniques des systèmes de composants, les réceptions sont dérivées, ce qui caractérise les propriétés mécaniques du système couplé entier. La nécessité de continuité du système aux points de couplage donne naissance à des conditions d'équilibre et de compatibilité aux connections. Ces conditions sont modifiées, permettant la présence d'unités de couplage élastiques et/ou dissipatives, ayant des masses négligeables entre les points de couplage, ajoutant donc une flexibilité pratique considérable à la méthode. En identifiant les contributions des composants des systèmes individuels, il est alors indiqué comment les réceptions entrent dans les calculs des résponses pour le système entier qui est sujet à des excitations déterminées ou hasardeuses.

Zusammenfassung—Eine Methode wird beschrieben für die Analyse komplexer Struktursysteme, die in eine Anzahl von Bestandteilsystemen aufgeteilt werden können, die willkührlich in bestimmten Punkten miteinander verbunden werden. Durch Verwendung experimentell oder analytisch ermittelter "Rezeptanzen" (Funktionen des Frequenzganges) die die mechanischen Eigenschaften des Bestandteilsystemes darstellen, werden Rezeptanzen ermittelt, die die mechanischen Eigenschaften des gekoppelten Gesamtsystems darstellen. Die Notwendigkeit der Systems-Kontinuität an den Kopplungsstellen gibt Anlass zu Gleichgewichtsbedingungen sowie zur Kompatibilität an den Verbindungsstellen. Diese Bedingungen sind abgeändert und gestatten die Anwesenheit elastischer sowie/oder zerstreuender Kopplungseinheiten mit sehr geringen Massen zwischen den Verbindungspunkten, dadurch erhält die Methode praktische Flexibilität in bedeutendem Mass. Indem man die Beiträge der einzelnen Bestandteilsysteme separat erkennbar hält wird dann gezeigt, dass und wie die Rezeptanzen in die Gesamtysstems-Berechnungen eintreten, wenn dies bestimmten oder willkührlichen Erregungen unterworfen wird.

Абстракт—Предлагается метод для анализа сложных структурных систем, которые могут быть подразделены на любое число компонентных систем, произвольно сопряжённых в дискретных пунктах. Применяя экспериментально или аналитически определённые приёмы (функции частотных характеристик)—характеризующие механические свойства компонентных систем, выводятся приёмы, которые характеризуют механические свойства всей спаренной системы. Требование непрерывности системы у соединительных пунктов повышают условия равновесия и совместимости в местах соединений. Эти условия изменяются, давая возможность присутствия эластически и/или диссипатически спаренных единиц с незначительными массами между пунктами спаривания, таким образом придавая методу значительную практическую гибкость. Придерживаясь отождествления содействия индивидуальных компонентных систем, указывается, как приёмы входят в частотные вычисления для всей системы, которая подчиняется определённым или случайным возбуждениям.